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Conservation Laws for Classical and Relativistic Collisions. II

EDWARD A. DESLOGE

Physics Department, Florida State University, Tallahassee, Florida 32306

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Abstract

On the basis of elementary symmetry arguments it is shown that (1) if in classical mechanics there exists a quantity $\lambda + \Sigma_i \mu_i v_i + \frac{1}{2}\nu v^2$ that is conserved, where λ, μ_i, and ν are particle parameters, then the μ_i and ν are all proportional to a single parameter μ and the quantity $A\mu + \Sigma_i B_i \mu v_i + C(\lambda + \frac{1}{2}D\mu v^2)$, where $D \equiv \nu/\mu$, is conserved for all values of A, B_i, and C; (2) if in relativistic mechanics there exists a quantity $\lambda + \Sigma_i \mu_i v_i[1 - (v^2/c^2)]^{-1/2} + \nu c[1 - (v^2/c^2)]^{-1/2}$ that is conserved, then the μ_i and ν are all proportional to a single parameter μ and the quantity $A\lambda + \Sigma_i B_i \mu v_i[1 - (v^2/c^2)]^{-1/2} + C\mu c[1 - (v^2/c^2)]^{-1/2}$ is conserved for all values of A, B_i, and C.

1. *Introduction*

In the preceding paper (this issue) it was shown that in a collision between two particles any conserved quantity, depending only on the nature and velocity of a particle, must classically be of the form

$$g = \lambda + \sum_i \mu_i v_i + \tfrac{1}{2}\nu v^2 \tag{1.1}$$

and relativistically of the form

$$g = \lambda + \sum_i \mu_i v_i[1 - (v^2/c^2)]^{-1/2} + \nu c[1 - (v^2/c^2)]^{-1/2} \tag{1.2}$$

where λ, μ_i, and ν are particle parameters

The proof of this statement was based on the assumption that any quantity that is conserved must be conserved in all inertial frames, and on the assumption that when two identical particles moving along a line with the same speed but in opposite directions collide, then it is possible for the same particles to emerge moving with their original speeds and in opposite directions but along a different line, and that the directions of the line of approach and the line of recession may assume any value.

In this paper I will show that if one assumes that a quantity that is conserved in one inertial frame is conserved in all inertial frames, and for every collision a collision in which the velocities of the incident and outgoing particles are either reflected in a plane or rotated through the same amount is a possible collision, then one can prove the following results.

Classical Collisions. If there exists a quantity

$$g = \lambda + \sum_i \mu_i v_i + \tfrac{1}{2}\nu v^2 \tag{1.3}$$

that is conserved in a collision, then we have the following:

(a) There exists a particle parameter μ and a set of universal constants B_1, B_2, B_3, and D such that

$$\mu_i = B_i\mu \tag{1.4}$$

$$\nu = D\mu \tag{1.5}$$

If at least one of the parameters μ_i, ν is not zero, then the ratio of the value of μ for one particle to the value of μ for a second particle is unique for the given pair of particles.

(b) The quantity

$$G = A\mu + \sum_i B_i\mu v_i + C(\lambda + \tfrac{1}{2}D\mu v^2) \tag{1.6}$$

is a conserved quantity not only for the particular values of A, B_i, C, and D for which G reduces to g, but for any other choice of the constants A, B_i, and C.

Relativistic Collisions. If there exists a quantity

$$g = \lambda + \sum_i \mu_i v_i[1-(v^2/c^2)]^{-1/2} + \nu c[1-(v^2/c^2)]^{-1/2} \tag{1.7}$$

that is conserved in a collision, then we have the following:

(a) There exists a particle parameter μ and a set of universal constants B_1, B_2, B_3, and C such that

$$\mu_i = B_i\mu \tag{1.8}$$

$$\nu = C\mu \tag{1.9}$$

If at least one of the parameters μ_i, ν is not zero, then the ratio of the value of μ for one particle to the value of μ for a second particle is unique for the given pair of particles.

(b) The quantity

$$G = A\lambda + \sum_i B_i\mu v_i[1-(v^2/c^2)]^{-1/2} + C\mu c[1-(v^2/c^2)]^{-1/2} \tag{1.10}$$

is a conserved quantity not only for the particular values of A, B_i, and C for which G reduces to g, but for any other choice of the constants A, B_i, and C.

In the following sections we will prove the above statements.

2. *Some Useful Theorems*

In later sections we will find the following two theorems useful. Both theorems are consequences of the definition of a conserved quantity.

> *Theorem 1.* If $g'(\mathbf{v})$ and $g''(\mathbf{v})$ are conserved quantities, then any linear combination of $g'(\mathbf{v})$ and $g''(\mathbf{v})$ is a conserved quantity, or equivalently
>
> $$g(\mathbf{v}) = Ag'(\mathbf{v}) + Bg''(\mathbf{v}) \tag{2.1}$$
>
> where A and B are arbitrary constants, is a conserved quantity.

Proof. The proof of this theorem follows immediately from the definition of a conserved quantity.

> *Theorem 2.* If there exists a conserved quantity of the form
>
> $$g(\mathbf{v}) = \alpha + \beta h(\mathbf{v}) \tag{2.2}$$
>
> where α and β are particle parameters and $h(\mathbf{v})$ is a universal function of the velocity other than a constant, and β is not identically zero, then the ratio of the value of β for one particle to the value of β for a second particle is unique for the given pair of particles.

Proof. Consider a collision in which two particles a and b moving with velocities $\mathbf{v}(a)$ and $\mathbf{v}(b)$, respectively, collide and end up with velocities $\mathbf{v}^*(a)$ and $\mathbf{v}^*(b)$, respectively. If $\alpha + \beta h(\mathbf{v})$ is a conserved quantity then

$$\alpha(a) + \beta(a)h[\mathbf{v}(a)] + \alpha(b) + \beta(b)h[\mathbf{v}(b)]$$
$$= \alpha(a) + \beta(a)h[\mathbf{v}^*(a)] + \alpha(b) + \beta(b)h[\mathbf{v}^*(b)] \tag{2.3}$$

If we let

$$\Delta h(a) \equiv h[\mathbf{v}^*(a)] - h[\mathbf{v}(a)] \tag{2.4}$$

$$\Delta h(b) \equiv h[\mathbf{v}^*(b)] - h[\mathbf{v}(b)] \tag{2.5}$$

then we can rewrite equation (2.3) as

$$\beta(a)\Delta h(a) + \beta(b)\Delta h(b) = 0 \tag{2.6}$$

Now suppose there exists a second constant of the motion $\alpha' + \beta' h(\mathbf{v})$. Then by the same reasoning we expect

$$\beta'(a)\Delta h(a) + \beta'(b)\Delta h(b) = 0 \tag{2.7}$$

Equations (2.6) and (2.7) constitute a set of simultaneous equations in the quantities $\Delta h(a)$ and $\Delta h(b)$, and will have a nontrivial solution if and only if

$$\beta(a)/\beta'(a) = \beta(b)/\beta'(b) \tag{2.8}$$

Since the particles a and b are arbitrary we conclude that in general for any particle β/β' is a universal constant. The theorem follows immediately.

3. *Basic Postulates*

The arguments in this paper will be based on the following postulates:

Postulate 1. If a quantity is conserved in one inertial frame, then it is conserved in all inertial frames.
Postulate 2. For every collision that occurs, a collision in which the velocities of the incident and outgoing particles are reflected in a plane is a possible collision.
Postulate 3. For every collision that occurs, a collision in which the velocities of the incident and outgoing particles are rotated the same arbitrary amount is a possible collision.

4. *Classical Collisions*

If we are given a conserved quantity $g(\mathbf{v})$ then we can generate other conserved quantities by making use of theorems 1 and 2 and the following theorems which can be derived from postulates 1, 2, and 3:

Theorem 3. If $g(\mathbf{v})$ is a conserved quantity then $g(-\mathbf{v})$ is a conserved quantity.
Theorem 4. If $g(v_i, v_j, v_k)$ is a conserved quantity where $i \neq j \neq k \neq i$ then $g(v_j, v_k, v_i)$ is a conserved quantity.
Theorem 5. If $g(v_i, v_j, v_k)$ is a conserved quantity where $i \neq j \neq k \neq i$ then $g(-v_i, v_j, v_k)$ is a conserved quantity.
Theorem 6. If $g(\mathbf{v})$ is a conserved quantity then $g(\mathbf{v} + \mathbf{V})$ is a conserved quantity for arbitrary constant $\mathbf{V}$.

Using Theorems 1-6 we can show that if there exists a conserved quantity

$$g = \lambda + \sum_i \mu_i v_i + \tfrac{1}{2}\nu v^2 \tag{4.1}$$

then the following quantities are also conserved quantities:

$$g_1 = \tfrac{1}{2}[g(\mathbf{v}) + g(-\mathbf{v})] = \lambda + \tfrac{1}{2}\nu v^2 \tag{4.2}$$

$$g_2 = \tfrac{1}{2}[g(\mathbf{v}) - g(-\mathbf{v})] = \sum_i \mu_i v_i \tag{4.3}$$

$$g_3 = \tfrac{1}{2}[g_2(v_i, v_j, v_k) - g_2(-v_i, v_j, v_k)] = \mu_i v_i \tag{4.4}$$

$$g_4 = g_3(v_j, v_k, v_i) = \mu_i v_j \tag{4.5}$$

$$g_5 = V_i^{-1}[g_3(\mathbf{v} + \mathbf{V}) - g_3(\mathbf{v})] = \mu_i \tag{4.6}$$

$$g_6 = g_1(\mathbf{v} + \mathbf{V}) - g_1(\mathbf{v}) = \nu \sum_i v_i V_i + \tfrac{1}{2}\nu V^2 \tag{4.7}$$

$$g_7 = V^{-2}[g_6(\mathbf{v}) + g_6(-\mathbf{v})] = \nu \tag{4.8}$$

$$g_8 = \tfrac{1}{2}[g_6(\mathbf{v}) - g_6(-\mathbf{v})] = \nu \sum_i v_i V_i \tag{4.9}$$

$$g_9 = (1/2V_i)[g_8(v_i, v_j, v_k) - g_8(-v_i, v_j, v_k)] = \nu v_i \tag{4.10}$$

Since $\mu_1 v_i, \mu_2 v_i, \mu_3 v_i$, and νv_i are conserved quantities, then by virtue of Theorem 2 there exists a set of constants B_1, B_2, B_3, and D and a particle parameter μ such that

$$\mu_i = B_i \mu \tag{4.11}$$

$$\nu = D\mu \tag{4.12}$$

Finally, from Theorem 1 any linear combination of the above quantities is a conserved quantity, and hence

$$G = A\mu + \sum_i B_i \mu v_i + C[\lambda + \tfrac{1}{2}D\mu v^2] \tag{4.13}$$

is a conserved quantity for any choice of the constants A, B_i, and C.

5. *Relativistic Collisions*

The same program that we followed for classical collisions can be followed for relativistic collisions. The mathematics can be simplified if instead of dealing with the velocities $\mathbf{v}$ and $\mathbf{V}$ we define the following quantities:

$$\mathbf{u} \equiv \gamma \mathbf{v} \tag{5.1}$$

$$\gamma \equiv [1 - (v^2/c^2)]^{-1/2} \equiv [1 + (u^2/c^2)]^{1/2} \tag{5.2}$$

$$\mathbf{U} \equiv \Gamma \mathbf{V} \tag{5.3}$$

$$\Gamma \equiv [1 - (V^2/c^2)]^{-1/2} \equiv [1 + (U^2/c^2)]^{1/2} \tag{5.4}$$

If we are given a conserved quantity $g(\mathbf{u})$ then we can generate other conserved quantities by making use of Theorems 1 and 2 and the following theorems which can be derived from postulates 1, 2, and 3:

> *Theorem 3'*. If $g(\mathbf{u})$ is a conserved quantity then $g(-\mathbf{u})$ is a conserved quantity.
> *Theorem 4'*. If $g(u_i, u_j, u_k)$ is a conserved quantity where $i \neq j \neq k \neq i$ then $g(u_j, u_k, u_i)$ is a conserved quantity.
> *Theorem 5'*. If $g(u_i, u_j, u_k)$ is a conserved quantity where $i \neq j \neq k \neq i$ then $g(-u_i, u_j, u_k)$ is a conserved quantity.
> *Theorem 6'*. If $g(\mathbf{u})$ is a conserved quantity then $g\{\mathbf{u} + \gamma\mathbf{U} + [(\Gamma - 1)(\mathbf{U}\cdot\mathbf{u})\mathbf{U}/U^2]\}$ is a conserved quantity.

Using Theorems 1, 2, 3'-6', we can show that if there exists a conserved quantity

$$g = \lambda + \sum_i \mu_i u_i + \nu(c^2 + u^2)^{1/2} \tag{5.5}$$

then the following quantities are also conserved:

$$g_1 = \tfrac{1}{2}[g(\mathbf{u}) + g(-\mathbf{u})] = \lambda + \nu(c^2 + u^2)^{1/2} \tag{5.6}$$

$$g_2 = \tfrac{1}{2}[g(\mathbf{u}) - g(-\mathbf{u})] = \sum_i \mu_i u_i \tag{5.7}$$

$$g_3 = \tfrac{1}{2}[g_2(u_i, u_j, u_k) - g_2(-u_i, u_j, u_k)] = \mu_i u_i \tag{5.8}$$

$$g_4 = g_3(u_j, u_k, u_i) = \mu_i u_j \tag{5.9}$$

$$\begin{aligned} g_5 &= g_3\{\mathbf{u} + \gamma\mathbf{U} + [(\Gamma - 1)(\mathbf{U}\cdot\mathbf{u})\mathbf{U}/U^2]\} - g_3(\mathbf{u}) \\ &= \mu_i\gamma U_i + \mu_i[(\Gamma - 1)(\mathbf{U}\cdot\mathbf{u})U_i/U^2] \end{aligned} \tag{5.10}$$

$$g_6 = (c/2U_i)[g_5(\mathbf{u}) + g_5(-\mathbf{u})] = \mu_i c\gamma = \mu_i(c^2 + u^2)^{1/2} \tag{5.11}$$

$$\begin{aligned} g_7 &= g_1\{\mathbf{u} + \gamma\mathbf{U} + [(\Gamma - 1)(\mathbf{U}\cdot\mathbf{u})\mathbf{U}/U^2]\} - g_1(\mathbf{u}) \\ &= \nu(\Gamma - 1)(c^2 + u^2)^{1/2} + \nu(\mathbf{U}\cdot\mathbf{u}/c) \end{aligned} \tag{5.12}$$

$$g_8 = [1/2(\Gamma - 1)]\,[g_7(\mathbf{u}) + g_7(-\mathbf{u})] = \nu(c^2 + u^2)^{1/2} \tag{5.13}$$

$$g_9 = \tfrac{1}{2}c\,[g_7(\mathbf{u}) - g_7(-\mathbf{u})] = \nu\mathbf{U}\cdot\mathbf{u} = \nu\sum_i U_i u_i \tag{5.14}$$

$$g_{10} = (1/2U_i)[g_9(u_i, u_j, u_k) - g_9(-u_i, u_j, u_k)] = \nu u_i \tag{5.15}$$

$$g_{11} = g_1 - g_8 = \lambda \tag{5.16}$$

Since $\mu_1 u_i, \mu_2 u_i, \mu_3 u_i$, and νu_i are conserved quantities, then by virtue of Theorem 2 there exists a set of constants B_1, B_2, B_3, and C and a particle parameter μ such that

$$\mu_i = B_i\mu \tag{5.17}$$

$$\nu = C\mu \tag{5.18}$$

The same result follows from the fact that $\mu_i(c^2 + u^2)^{1/2}$ and $\nu(c^2 + u^2)^{1/2}$ are conserved quantities.

Finally, from Theorem 1 any linear combination of the above quantities is a conserved quantity, and hence

$$G = A\lambda + \sum_i B_i\mu u_i + C\mu(c^2 + u^2)^{1/2} \tag{5.19}$$

is a conserved quantity for any choice of the constants A, B_i, and C.

6. *Conclusion*

In this paper and the preceding paper we have shown that the possible strictly velocity-dependent conserved quantities are extremely limited.

The results obtained have interesting implications for the logical structure of classical and relativistic mechanics. One can show, for example, that classical dynamics can be built on the assumption that there exists a quantity μv_i that is conserved, and relativistic dynamics can be built on the assumption that there exists a quantity μu_i that is conserved. In relativistic dynamics if any velocity-dependent quantity is conserved then μu_i is conserved, and if μu_i is conserved then all possible velocity-dependent quantities that could be conserved are conserved, where we are excluding conserved quantities of the form $g = \lambda$. This suggests that the basic postulate of relativistic dynamics might be stated simply as: "Any quantity which can be conserved is conserved." A similar formulation could be devised for classical dynamics.